On highly regular strongly regular graphs

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**$k$-Homogeneity**

The local-global principle

Every isomorphism between two substructures of a structure can be extended to an automorphism of the structure.

The formal definition for graphs

Let $\Gamma = (V, E)$ be a graph. Gamma is called $k$-homogeneous if for all $V_1, V_2 \subseteq V$ with $|V_1| = |V_2| \leq k$ and for each isomorphism $\psi : \Gamma(V_1) \rightarrow \Gamma(V_2)$ there exists an automorphism $\varphi$ of $\Gamma$ such that $\varphi|_{V_1} = \psi$. 
Types

Category of Graphs

- A graph-homomorphism \( h : \Gamma_1 \rightarrow \Gamma_2 \) is a function from \( V(\Gamma_1) \) to \( V(\Gamma_2) \) that maps edges to edges.
- an embedding is an injective homomorphism with the property that the preimage of edges are edges, too.

Regularity-Types

- A regularity-type (or type, for short) is an embedding of finite graphs.
- Regularity-types are denoted like \( \mathcal{T} : \Gamma_1 \hookrightarrow \Gamma_2 \).

Order of a Type

\( \mathcal{T} : \Gamma_1 \hookrightarrow \Gamma_2 \) has order \((n, m)\) if \( |V(\Gamma_1)| = n \), and \( |V(\Gamma_2)| = m \)
**T-regularity**

**Given:**
- A graph $\Gamma$,
- a type $T : \Delta_1 \rightsquigarrow \Delta_2$

**Counting $T$:**
- Let $\iota : \Delta_1 \mapsfrom \Gamma$.
- $\#(\Gamma, T, \iota)$ we define to be the number of embeddings $\hat{\iota} : \Delta_2 \mapsfrom \Gamma$ that make the following diagram commute:
\( \mathbb{T} \)-regularity (cont.)

\( \mathbb{T} \)-regularity

\( \Gamma \) is called \( \mathbb{T} \)-regular if the number \( \#(\Gamma, \mathbb{T}, \iota) \) does not depend on the embedding

\[ \iota : \Delta_1 \hookrightarrow \Gamma. \]

In this case this number is denoted by \( \#(\Gamma, \mathbb{T}) \)

Remark

A concept equivalent to \( \mathbb{T} \)-regularity, but in the category of complete colored graphs, was introduced and studied by Evdokimov and Ponomarenko (2000) in relation with the \( t \)-vertex condition for association schemes.

Example

If \( \mathbb{T} \) is given by

\[ \begin{array}{c}
\circ \\
\times
\end{array} \begin{array}{c}
\rightarrow
\end{array} \begin{array}{c}
\circ \\
\times
\end{array} \]

then \( \Gamma \) is \( \mathbb{T} \)-regular if and only if it is regular.
\((n, m)\)-regularity

**Definition**

A graph \( \Gamma \) is \((n, m)\)-regular if for all \( 1 \leq k \leq n \) and \( k < l \leq m \), and for every type \( \mathbb{T} \) of order \((k, l)\) we have that \( \Gamma \) is \( \mathbb{T} \)-regular.

- \((1, 2)\)-regular is the same as regular,
- \((2, 3)\)-regular is the same as strongly regular,
- \((k, k + 1)\)-regular is the same as \(k\)-isoregular,
- \((2, t)\)-regular is the same as fulfilling the \(t\)-vertex condition.
Known examples

Hestenes, Higman (1971): Point graphs of generalized quadrangles fulfill the 4-vc,

A.V.Ivanov (1989): found a graph on 256 vertices with the 4-vc (not 2-homogeneous),

Brouwer, Ivanov, Klin (1989): generalization to an infinite series,

A.V.Ivanov (1994): another infinite series of graphs with the 4-vc,

Reichard (2000): both series fulfill the 5-vc,

A.A.Ivanov, Faradžev, Klin (1984) constructed a srg on 280 vertices with Aut($J_2$) as automorphism group,

Reichard (2000): this graph fulfills the 4-vc,

Reichard (2003): point graphs of GQ($s, t$) fulfill the 5-vc,

Reichard (2003): point graphs of GQ($q, q^2$) fulfill the 6-vc,

Klin, Meszka, Reichard, Rosa (2003): the smallest srgs with 4-vc have parameters $(36, 14, 4, 6),$

CP (2004): point graphs of partial quadrangles fulfill the 5-vc,

Reichard (2005): point graphs of GQ($q, q^2$) fulfill the 7-vc,

CP (2007): point graphs of PQ($q - 1$, $q^2$, $q^2 - q$) fulfill the 6-vc,

Klin, CP (2007): found two self-complementary graphs that fulfill the 4-vc.
Counting types in graphs is algorithmically hard.
Luckily, in general, it is not necessary to count all types.

Example (Hestenes-Higman-Theorem)
In order to prove that a graph fulfills the 4-vertex condition for a graph, it is enough to prove that it is $T$-regular for the following types:

- $x - y$
- $x - y - y$
- $x - x - y$
- $x - x - y - y$
- $x - x - x - y$
- $x - x - x - y - y$
Composing types

Given:

\( T_1 : \Delta_1 \leftrightarrow \Delta_2, \ T_2 : \Delta_3 \leftrightarrow \Delta_4, \ e : \Delta_3 \leftrightarrow \Delta_2. \)

Consider the following diagram:

\[
\begin{array}{c}
\Delta_4 \\
\uparrow \quad T_2 \\
\Delta_2 \\
\quad \quad e \\
\quad \quad \downarrow \quad T_1 \\
\Delta_3 \\
\downarrow \quad \downarrow \\
\Delta_1
\end{array}
\]
Composing types

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Consider the following diagram:

Let \( \Lambda \) be a colimes.
Composing types

Given:

\( T_1 : \Delta_1 \leftrightarrow \Delta_2, \ T_2 : \Delta_3 \leftrightarrow \Delta_4, \ e : \Delta_3 \leftrightarrow \Delta_2. \)

Consider the following diagram:

\[ \begin{array}{c}
\Lambda & \leftarrow & \Delta_2 \leftarrow & \Delta_3 \rightarrow & \Delta_1 \\
& \leftarrow & ^{\iota_1} & e & \\
& & \rightarrow & \Delta_4 \\
\end{array} \]

Let \( \Lambda \) be a colimes.

Then \( \iota_1 \) is a type. It is denoted by \( T_1 \oplus_e T_2. \)
Comparison of Types

Given:
\[ T_1 : \Delta_1 \leftrightarrow \Delta_2, \ T_2 : \Delta_1 \leftrightarrow \Delta_3. \]

Definition
We define \( T_1 \preceq T_2 \) if there is an epimorphism \( \tau : \Delta_2 \twoheadrightarrow \Delta_3 \) that makes the following diagram commute:

\[
\begin{array}{c}
\Delta_2 \\
\downarrow \tau \\
\Delta_1 \end{array}
\begin{array}{c}
\Delta_3 \\
\uparrow T_2 \\
\end{array}
\begin{array}{c}
T_1 \\
\end{array}
\]

If \( \tau \) is not an isomorphism, then we write \( T_1 \prec T_2 \).
Type-Counting Lemma

Given:
- $T_1 : \Delta_1 \rightarrow \Delta_2$, $T_2 : \Delta_3 \rightarrow \Delta_4$, $e : \Delta_3 \rightarrow \Delta_2$,
- a graph $\Gamma$.

Lemma

If $\Gamma$ is $T_1$- and $T_2$-regular, and if $\Gamma$ is $T$-regular for all $T_1 \oplus_e T_2 \prec T$, then $\Gamma$ is also $T_1 \oplus_e T_2$-regular.
Definition

Let $\mathbb{T} : \Delta \leftrightarrow \Theta$ be a regularity-type of order $(m, n)$. Suppose $\Delta = (B, D)$, $\Theta = (T, E)$. Let $S \subseteq T$ be the image of $\mathbb{T}$. Then we define

$$\hat{T} := (T, E \cup (S/2)),$$

Proposition

Let $\Gamma$ be an $(n, m)$-regular graph (for $m > n$). Then, $\Gamma$ is $(n, m + 1)$-regular if and only if it is $\mathbb{T}$-regular for all graph-types $\mathbb{T}$ of order $(n, m + 1)$ for which $\hat{T}$ is $(n + 1)$-connected.
Definition

An incidence structure is a triple \((\mathcal{P}, \mathcal{L}, \mathcal{I})\) such that

1. \(\mathcal{P}\) is a set of points,
2. \(\mathcal{L}\) is a set of lines,
3. \(\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}\).

Definition

A partial linear space of order \((s, t)\) is an incidence structure \((\mathcal{P}, \mathcal{L}, \mathcal{I})\) such that

1. each line is incident with \(s + 1\) points,
2. each point is incident with \(t + 1\) lines,
3. every pair of points is incident with at most one line.

Definition

The point graph of a partial linear space is the graph that has as vertices the points such that two points form an edge whenever they are collinear.
Partial Linear spaces

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Definition

The point graph of a partial linear space is the graph that has as vertices the points such that two points form an edge whenever they are collinear.
Definition (Cameron 1975)

A partial quadrangle of order \((s, t, \mu)\) is a partial linear space of order \((s, t)\) such that

1. if three points are pairwise collinear, then they are on one line,
2. if two points are non-collinear, then exactly \(t\) points are collinear with both.

Remarks

- A strongly regular graph is isomorphic to the point-graph of a PQ if and only if it does not contain a subgraph isomorphic to \(K_4 - e\) (Cameron ’75).
- Thus, the original PQ can be recovered from its point graph, up to isomorphism.
- Without loss of generality, we may identify a partial quadrangle with its point graph.
Generalized quadrangles

Definition

A *generalized quadrangle* of order $(s, t)$ is a partial linear space of order $(s, t)$ such that for every line $l$ and every point $P$ not on $l$ there is a unique point $Q$ on $l$ that is collinear with $P$.

Remark

*Every generalized quadrangle is also a partial quadrangle. Thus we may also identify a generalized quadrangle with its point graph.*

Proposition (Higman 1971)

A generalized quadrangle has order $(s, s^2)$ if and only if every triad (i.e. triple of pairwise non-collinear points) has the same number of centers.

In a $GQ(s, s^2)$ every triad has $s + 1$ centers.

Corollary

The point-graph of a $GQ(s, s^2)$ is $(3, 4)$-regular.
Generalized quadrangles

**Definition**

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**Proposition (Higman 1971)**

A generalized quadrangle has order \((s, s^2)\) if and only if every triad (i.e. triple of pairwise non-collinear points) has the same number of centers.  
In a GQ\((s, s^2)\) every triad has \(s + 1\) centers.

**Corollary**

The point-graph of a GQ\((s, s^2)\) is \((3, 4)\)-regular.
Step 1. Take a generalized quadrangle $(P, L, I)$ of order $(s, s^2)$.

Step 2. Take some point $V \in P$ and define

$$P_V := \{P \in P \mid P \neq V, \, P, \, V \text{ not collinear}\},$$

$$L_V := \{l \in L \mid V \text{ not on } l\},$$

$$I_V := I \cap (P_V \times L_V).$$

Proposition

*The incidence structure $(P_V, L_V, I_V)$ is a partial quadrangle of order $(s - 1, s^2, s^2 - s)$*

Remarks

- In fact, every such $PQ(s - 1, s^2, s^2 - s)$ has an extension to a $GQ(s, s^2)$ (Ivanov, Shpektorov 1991).
- Thus, every partial quadrangle of order $(s - 1, s^2, s^2 - s)$ can be obtained this way.
On the (2, 6)-regularity of PQ(s − 1, s^2, s^2 − s)

- Proving (2, 6)-regularity of PQ(s − 1, s^2, s^2 − s) so far made heavily use of the (2, 7)-regularity and the (3, 4)-regularity of the associated GQ(s, s^2).
- A close analysis of the (very technical) proof revealed that in fact for many types T of order (3, 7), it was shown that GQ(s, s^2) is T-regular.

Goal:
Simplify the proof of the (2, 6)-regularity of PQ(s − 1, s^2, s^2 − s) by studying types of order (3, 7) in the associated GQ(s, s^2).
Main result

Question
Which types have to be counted in order to show that a GQ\((s, s^2)\) is \((3, 7)\)-regular?

Answer
Only the types \(T_1 : K_3 \leftrightarrow K_5, \; T_2 : K_3 \leftrightarrow K_6, \; T_3 : K_3 \leftrightarrow K_7\) need to be counted.

Theorem
GQ\((s, s^2)\) is \((3, 7)\)-regular.

Proof.
Needed in the proof:
1. \((3, 4)\)-regularity of GQ\((s, s^2)\),
2. the type-counting lemma,
3. a computer for finding all indecomposable types.
Concluding remarks

Consequences

1. The presented result strengthens Reichard’s theorem on the 7-vertex condition for \( GQ(s, s^2) \).
2. The proof of the presented result drastically simplifies the proof of the 6-vertex condition for \( PQ(s - 1, s^2, s^2 - s) \).

Smallest non-classical \((3, 7) – regular\) example

- The smallest non-classical example is \( GQ(5, 25) \).
- Its point-graph has parameters \((v, k, \lambda, \mu) = (756, 130, 4, 26)\).
- Its automorphism group acts intransitively on the vertices.
Conjecture (Klin 1994?)

There exists a $t_0$, such that every strongly regular graph that satisfies the $t_0$-vertex condition is in fact already 2-homogeneous (i.e., is a rank-3-graph).

- if Klin's conjecture is true, then $t_0 \geq 8$ (by Reichard's result).
- Moreover, if a Moore-graph of valency 57 exists, then $t_0 \geq 10$ (Reichard, CP 2014).

Problem

Does there exist a $t_0$, such that every $(3, t_0)$-regular graph is 3-homogeneous?

If so, then $t_0 \geq 8$.

Problem

Does there exists a $t_0$, such that every $(4, t_0)$-regular graph is 4-homogeneous?

If so, then $t_0 \geq 6$, as the only known $(4, 5)$-regular graph that is not 4-homogeneous is the McLaughlin-graph (on 275 vertices). It is not $(4, 6)$-regular.