Two-graphs revisited

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Modern Trends in Algebraic Graph Theory

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The rotation group of the icosahedron (which is isomorphic to $A_5$) permutes the six diagonals 2-transitively and preserves the two types of triples.
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A little earlier, motivated by questions in elliptic geometry, specifically the congruence order of the elliptic plane, Jaap Seidel had represented a set of \( n \) equiangular lines by a graph on \( n \) vertices, or more precisely, a **switching class** of graphs. (The operation of switching with respect to a set of vertices interchanges edges and non-edges between the set and its complement, while leaving edges within the set or within its complement unchanged.)
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The Seidel tree in Eindhoven
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Seidel wrote a number of surveys of two-graphs in the 1970s and 1980s. Recently there have been some new developments. I will briefly describe the equivalences and then discuss some new things.
Two-graphs and switching classes

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A consequence is that the automorphism groups of the two-graph and of the corresponding switching class of graphs are equal. (An automorphism of a switching class is a vertex permutation which carries some, and hence every, graph in the class to another graph in the class.) This group contains the automorphism group of each graph in the switching class as a subgroup.
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**Theorem**

*Given any finite group $G$, there is a switching class $\mathcal{C}$ of graphs with the properties*

- $\text{Aut}(\mathcal{C}) = G$;
- *for each subgroup $H \leq G$, there is a graph $\Gamma \in \mathcal{C}$ with $\text{Aut}(\Gamma) = H$.***
A binary representation

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These facts (easily shown directly) state that the $\mathbb{F}_2$ cohomology of the simplex vanishes in dimensions 0, 1 and 2.
Double covers of complete graphs

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Equiangular lines

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Switching the graph corresponds to pre- and post-multiplying $A$ by a diagonal matrix with entries $-1$ on the switching set and $+1$ elsewhere. So graphs in the same switching class have Seidel adjacency matrices which are similar, and so have the same **Seidel spectrum**.
Since $A$ has trace 0, its smallest eigenvalue is negative, say $-\lambda$, with multiplicity $n - d$, say. (Assume that the graph is not null.) Then $A + \lambda I$ is positive semi-definite, and so is the Gram matrix of inner products of a set of vectors in $\mathbb{R}^d$. Each vector has squared length $\lambda$, and the inner products of different vectors are $\pm 1$; so the angle between any two vectors is $\arccos(1/\lambda)$ or $\pi - \arccos(1/\lambda)$. Thus the lines spanned by these vectors are equiangular.
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Conversely, given a set of equiangular lines, choose a unit vector along each line as a vertex of a graph, where two vertices are adjacent if the vectors make an acute angle. Different choices of vectors give switching-equivalent graphs.
Acute angles

Obtuse angles
Heavily covered points

The following theorem was proved by Boros and Füredi for $d = 2$, and Bárány for arbitrary $d$.

**Theorem**

For any $d \geq 2$, there is a positive constant $c_d$ with the property that, for any set of $n$ points in general position in $\mathbb{R}^d$, there is a point (not necessarily one of the given points), which lies in a proportion at least $c_d$ of the $\binom{n}{d+1}$ simplexes spanned by the $(d + 1)$-sets of points in the set.
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The correct value of $c_2$ (the supremum of all real numbers for which the statement holds) is $2/9$; but for $d > 2$, only upper and lower bounds are known.
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The density of a graph (or a two-graph) is the proportion of all 2-sets (resp. 3-sets) which are edges of the graph (resp. triples of the two-graph. Gromov’s function $\phi_2$ is defined by the rule that $\phi_2(\alpha)$ is the limit inferior of the densities of two-graphs with the property that all graphs in the corresponding switching class have density at least $\alpha$. 
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In this way, they improved the lower bound on $c_3$ from 0.06332 to 0.07480. (The best known upper bound is 0.09375.)
Statistics of switching classes

Computing Gromov’s $\phi_2$ is equivalent to finding the smallest number of edges in a graph in the switching class of a two-graph with given density. Note that the minimising graph has the property that, given any 2-partition of the vertices, at most half of the pairs crossing the partition are edges of the graph.
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More general question about the distribution of numbers of edges in the switching class of a graph could be asked. I will just make the easy remark that the average edge-density in any switching class is $1/2$, since half of the switching partitions separate any given pair of points.
Primitive groups are small

A permutation group $G$ acting on a set $X$ of $n$ points (with $n > 2$) is primitive if there is no partition of $X$ invariant under $G$ apart from the two trivial partitions (into singletons, and with a single part).
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In the early 1980s, with Peter Neumann and Jan Saxl, I proved:

**Theorem**

If $G$ is a primitive group on $X$, other than $S_n$ and $A_n$ and finitely many exceptions, then there is a subset of $X$ whose setwise stabiliser in $G$ is the identity.
Extensions

This result has been quantified in various ways:

- Akos Seress found the finitely many exceptions: there are 43 of them, the largest degree being 32.
- I showed that the proportion of subsets whose stabilizer is trivial tends to 1 as $n \to \infty$ (in primitive groups of degree $n$ other than $S_n$ and $A_n$).
- Laci Babai and I showed that we can take the size of the subset to be at most $n^{1/2} + o(1)$. 
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Here for the record is Seress’ list of primitive groups with no regular orbit on the power set, excluding symmetric and alternating groups. The first number is the degree and the second is the number of the group in the GAP library of primitive groups.

- \((5, 2, D_{10})\)
- \((5, 3, F_{20})\)
- \((6, 1, A_5)\)
- \((6, 2, S_5)\)
- \((7, 4, F_{42})\)
- \((7, 5, L_3(2))\)
- \((8, 2, 2.3.3.7.3)\)
- \((8, 3, L(2))\)
- \((8, 4, L_3(2).2)\)
- \((8, 5, 2.4.3.3.3)\)
- \((9, 2, 3.2.D_8)\)
- \((9, 5, 3.2.8.2)\)
- \((9, 6, 3.2.2.L_2(3))\)
- \((9, 7, 3.2.2.L_3(3) .2)\)
- \((9, 8, L_2(8))\)
- \((9, 9, L_2(8).3)\)
- \((10, 2, S_5)\)
- \((10, 3, A_6)\)
- \((10, 4, S_6)\)
- \((10, 5, A_6.2)\)
- \((10, 6, A_6.2)\)
- \((10, 7, A_6.2.2)\)
- \((11, 5, L_2(11))\)
- \((11, 6, M_{11})\)
- \((12, 2, L_2(11).2)\)
- \((12, 3, M_{11})\)
- \((12, 4, M_{12})\)
- \((13, 7, L_3(3))\)
- \((14, 2, L_2(13).2)\)
- \((15, 4, A_8)\)
- \((16, 16, 2.4.(A_5 \times 3).2)\)
- \((16, 17, 2.4.A_6)\)
- \((16, 18, 2.4.S_6)\)
- \((16, 19, 2.4.A_7)\)
- \((16, 20, 2.4.L_4(2))\)
- \((17, 7, L_2(16).2)\)
- \((17, 8, L_2(16).4)\)
- \((21, 7, L_3(4).3.2)\)
- \((22, 1, M_{22})\)
- \((22, 2, M_{22}.2)\)
- \((23, 5, M_{23})\)
- \((24, 3, M_{24})\)
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Theorem
Apart from the alternating groups and finitely many others, every primitive group is the full automorphism group (acting on vertices) of an edge-transitive hypergraph.
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A theorem about switching classes

I will present a new result along these lines.

Apart from the switching classes of the complete and null graphs, and finitely many others, every switching class with primitive automorphism group contains a graph with trivial automorphism group. Work to determine the finitely many exceptions is in progress.
A theorem about switching classes

I will present a new result along these lines. It might be thought that very symmetric switching classes (say those with primitive automorphism groups) will be made up of very symmetric graphs. But in fact, we have:

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Note that a graph basis is a base for the automorphism group of \( \Gamma \) (its pointwise stabiliser is trivial), so the graph dimension is an upper bound for the base size.
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Similarly, given a 3-uniform hypergraph $\Gamma$, a **hypergraph basis** for $\Gamma$ is a set $S$ of vertices such that, for distinct vertices $v \notin S$, the graphs $\Gamma_v(S)$ with edges the pairs $\{x,y\} \subseteq S$ for which $\{v,x,y\}$ is an edge of $\Gamma$, are distinct.
Descendants

If $T$ is a two-graph with a vertex $v$, there is a unique graph $\Gamma_v$ in the corresponding switching class which has $v$ as an isolated vertex. (Take any graph in the switching class, and switch with respect to the neighbours of $v$.) The graphs $\Gamma_v - v$ are called descendants of $T$. 

In particular, if two descendants are isomorphic, then the corresponding vertices are in the same orbit of the automorphism group of the two-graph; so if all descendants are isomorphic, then the automorphism group is transitive (and conversely).
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In particular, if two descendants are isomorphic, then the corresponding vertices are in the same orbit of the automorphism group of the two-graph; so if all descendants are isomorphic, then the automorphism group is transitive (and conversely).
Regularity

Recall that a two-graph is regular if any two vertices lie in the same number of triples. Now the following are equivalent:

- the two-graph $T$ is regular;
- the corresponding double cover is a Taylor graph (an antipodal distance-regular graph with diameter 3 and antipodal classes of size 2);
- some (or equivalently every) descendant of $T$ is strongly regular, with $k = 2$,
- the Seidel spectrum has just two eigenvalues.

For example, the descendants of the two-graph associated with the diagonals of the icosahedron are pentagons (as can be seen by looking along a diagonal). The descendants of the Higman two-graph for $Co_3$ are isomorphic to the McLaughlin strongly regular graph.
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**Theorem**

- The graph dimension of any graph in the switching class associated with a two-graph $T$ does not exceed the hypergraph dimension of $T$. 
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**Theorem**

- The graph dimension of any graph in the switching class associated with a two-graph $T$ does not exceed the hypergraph dimension of $T$.

- Let $v$ be a vertex of a two-graph $T$. Then the hypergraph dimension of $T$ is at most one more than the graph dimension of the descendant $\Gamma_v - v$. 