Spectral characterizations of distance-regularity of graphs

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A (finite simple) graph $\Gamma$ on $n$ vertices

The spectrum (of eigenvalues) $\lambda_1 \geq \ldots \geq \lambda_n$ of the (a) 01-adjacency matrix $A$ of $\Gamma$
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EvD & Haemers (2003) ‘would bet’ that almost all graphs are determined by the spectrum.
Distance-regularity: there are $c_i, a_i, b_i$, $i = 0, 1, \ldots, d$ such that for every pair of vertices $u$ and $w$ at distance $i$:

- The number of neighbors $z$ of $w$ at distance $i - 1$ from $u$ equals $c_i$.
- The number of neighbors $z$ of $w$ at distance $i$ from $u$ equals $a_i$.
- The number of neighbors $z$ of $w$ at distance $i + 1$ from $u$ equals $b_i$. 
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Fon-Der-Flaass (2002) $\Rightarrow$ Almost all distance-regular graphs are not determined by the spectrum.
Walks

\( A_i \) is the distance-\( i \) adjacency matrix, \( A = A_1 \):

\[
AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}, \quad i = 0, 1, \ldots, d,
\]

\( A_i = p_i(A) \) for a polynomial \( p_i \) of degree \( i \)

Rowlinson (1997): A graph is a DRG iff the number of walks of length \( \ell \) from \( x \) to \( y \) depends only on \( \ell \) and the distance between \( x \) and \( y \)
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Rowlinson (1997): A graph is a DRG iff the number of walks of length $\ell$ from $x$ to $y$ depends only on $\ell$ and the distance between $x$ and $y$.

Intersection numbers do not determine the graph (in general).

Do the eigenvalues determine distance-regularity?
\[
\sum (A^\ell)_{uu} = \text{tr } A^\ell = \sum \lambda_i^\ell
\]

\[
\sum p(A)_{uu} = \text{tr } p(A) = \sum p(\lambda_i)
\]

for every polynomial \( p \)

All spectral information is in these equations.
The following can be derived from the spectrum:

- number of vertices
- number of edges
- number of triangles
- number of closed walks of length $\ell$
- bipartiteness
- regularity
- regularity + connectedness
- regularity + girth
- odd-girth
Distance-regularity is not determined by the spectrum

The (‘almost’ dr) twisted Desargues graph
(Bussemaker & Cvetković 1976, Schwenk 1978)

Note: Desargues is Doubled Petersen
If \( \Gamma \) is distance-regular, diameter \( d \), valency \( k \), girth \( g \), distinct eigenvalues \( k = \theta_0, \theta_1, \ldots, \theta_d \), satisfying one of the following properties, then every graph cospectral with \( \Gamma \) is also distance-regular:

1. \( g \geq 2d - 1 \) (Brouwer & Haemers 1993),
2. \( g \geq 2d - 2 \) and \( \Gamma \) is bipartite (EvD & Haemers 2002),
3. \( g \geq 2d - 2 \) and \( c_{d-1}c_d < -(c_{d-1} + 1)(\theta_1 + \ldots + \theta_d) \) (EvD&Haemers 2002),
4. \( c_1 = \ldots = c_{d-1} = 1 \) (EvD & Haemers 2002),
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Moreover, the following graphs are determined by their spectrum:

1. dodecahedron and icosahedron (Haemers & Spence 1995),
2. coset graph extended ternary Golay code (EvD & Haemers 2002),
3. Ivanov-Ivanov-Faradjev graph (EvD & Haemers & Koolen & Spence 2006),
4. Hamming graph $H(3, q)$, $q \geq 36$ (Bang &EvD & Koolen 2008).
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Note: the Johnson graph $J(n, d)$, $n - 3 \geq d \geq 3$ has cospectral graphs that are not distance-regular \text{(EvD & Haemers & Koolen & Spence 2006)}.
EvD & Koolen & Tanaka, in preparation:

“Distance-regular graphs”
Consider the spectrum of a $k$-regular graph

**Inner product** $\langle p, q \rangle = \frac{1}{n} \text{tr}(p(A)q(A)) = \frac{1}{n} \sum_i p(\lambda_i)q(\lambda_i)$

on the space of polynomials mod minimal polynomial
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Orthogonal system of **predistance polynomials** $p_i$ of degree $i$ normalized such that $\langle p_i, p_i \rangle = p_i(k) \neq 0$

$x p_i = \beta_{i-1} p_{i-1} + \alpha_i p_i + \gamma_{i+1} p_{i+1}, \quad i = 0, 1, \ldots, d,$

compare to

$A A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}, \quad i = 0, 1, \ldots, d,$
Spectral Excess

Spectral Excess Theorem (Fiol & Garriga 1997):

\[ k_d \leq p_d(k) \] with equality iff the graph is distance-regular
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Laplacian Spectral Excess Theorem (EvD & Fiol 2014):

it is not necessary to restrict to regular graphs!
Preintersection numbers

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Preintersection numbers

Abiad & EvD & Fiol (2014) A non-bipartite graph has odd-girth $2m + 1$ if and only if $\alpha_0 = \cdots = \alpha_{m-1} = 0$ and $\alpha_m \neq 0$. A graph is bipartite if and only if $\alpha_0 = \cdots = \alpha_d = 0$. 

Villanova, June 2, 2014
Generalized Odd graph (drg with $a_1 = \ldots = a_{d-1} = 0, \ a_d \neq 0$)

Odd graphs, folded cubes, ‘almost bipartite’, $\{7, 6, 6; 1, 1, 2\}$
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EvD & Haemers (Odd-girth theorem 2011) A regular graph with $d + 1$ distinct eigenvalues and odd-girth $2d + 1$ is a generalized Odd graph

Lee & Weng (2012) extended this for non-regular graphs
Odd-girth

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Abiad & EvD & Fiol (2014) Let $G$ be non-bipartite graph with $\alpha_i \geq 0$ for $i = 0, \ldots, d - 1$. Then

$$\gamma_d \geq -(\theta_1 + \ldots + \theta_d),$$

with equality if and only if $G$ is a distance-regular generalized Odd graph.
Abiad & EvD & Fiol (2014):

A regular graph has girth $2m + 1$ if and only if $\alpha_0 = \cdots = \alpha_{m-1} = 0$, $\alpha_m \neq 0$, and $\gamma_1 = \cdots = \gamma_m = 1$.

A regular graph has girth $2m$ if and only if $\alpha_0 = \cdots = \alpha_{m-1} = 0$, $\gamma_1 = \cdots = \gamma_{m-1} = 1$, and $\gamma_m > 1$. 
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$\Gamma$ is distance-regular if any of the following conditions holds:

1. $g \geq 2d - 1$,
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3. $g \geq 2d - 2$ and $\gamma_d < -(\theta_1 + \cdots + \theta_d)$,
4. $\gamma_1 = \cdots = \gamma_{d-1} = 1$. 
Thanks

Thanks to

Aida Abiad
Sejeong Bang
Cristina Dalfó
Miquel Angel Fiol
Willem Haemers
Jack Koolen
Ted Spence
Hajime Tanaka
Zheng-jiang Xia
Fix a \((2d - 1)\)-set

Partial linear space of \((d - 1)\)-sets (points) vs. \(d\)-sets (lines)

Point graph and line graph are \(J(2d - 1, d - 1) \sim J(2d - 1, d)\)

Incidence graph is Doubled Odd graph
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Incidence graph is cospectral to Doubled Odd, but not distance-regular: twisted Doubled Odd

EvD & Haemers & Koolen & Spence (2006): \(DO(d), d \geq 3\) has one non-distance-regular cospectral graph that has (at least) one of the halved graphs isomorphic to \(J(2d - 1, d - 1)\).
Generalize to Grassmann graphs

Subsets $\rightarrow$ subspaces of a $(2d - 1)$-dimensional space over $GF(q)$
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Spectral excess theorem: line graph is distance-regular!

....but it is UGLY!!! (the twisted Grassmann graph)
Families of ‘ugly’ distance-regular graphs with unbounded diameter:

Doob, Hemmeter, Ustimenko: not distance-transitive.

twisted Grassmann (EvD & Koolen 2005): not even vertex-transitive.
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Note: the Grassmann graph $J_q(n, d), n - 3 \geq d \geq 3, q$ a prime power has cospectral graphs that are not distance-regular (EvD & Haemers & Koolen & Spence 2006).
Dalfó & EvD & Fiol (2011): Ugly (almost) distance-regular graphs can be used to construct cospectral graphs through perturbations:

Adding and removing vertices, edges, amalgamating vertices, etc.
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Removing vertices from the twisted **Desargues** graph

“It is easy to construct cospectral graphs from ugly d-r graphs”
EvD & Koolen & Xia (2014): “It is easy to construct cospectral graphs from BEAUTIFUL (distance-regular) Taylor graphs”