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Title: *Friendship Theorem for Hypergraphs: a Construction*

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A graph is a **friendship graph** if every two vertices have exactly one common neighbour.

**The Friendship Theorem** (Erdős, Rényi and Sós, 1966)

Every friendship graph has a universal friend, i.e., a vertex adjacent to every other vertex.

The edges of a friendship graph are partitioned in triangles.
An $r$-uniform hypergraph consists of a vertex set $V$ and set $E$ of hyperedges,
where a hyperedge is an $r$-subset of $V$.

A graph is a 2-uniform hypergraph.

**Definition** Sós, 1976 (for $r = 3$)
An $r$-uniform hypergraph is a **friendship hypergraph** if for any $r$-subset $S \subseteq V$ there is a unique vertex $w$ (called the completion of $S$) so that for every $v \in S$,

$$(S \setminus \{v\}) \cup \{w\} \in E.$$  

A friendship graph is a 2-uniform friendship hypergraph.
A vertex $\infty$ in an $r$-uniform hypergraph is called a universal friend if \( \{\infty, v_1, \ldots, v_{r-1}\} \) is a hyperedge for all \( v_1, \ldots, v_{r-1} \in V \setminus \{\infty\} \).

**Theorem**  Sós, 1976
A 3-uniform hypergraph $G$ with a universal friend $\infty$ is a friendship hypergraph if and only if $G - \infty$ is a Steiner triple system.

There is an obvious generalization of this theorem to $r \geq 4$. 
A **quad** or a $K_4^3$ in a 3-uniform hypergraph is a set $\{a, b, c, d\}$ so that $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ all are hyperedges.

The hyperedges of a 3-uniform friendship hypergraph are partitioned in quads.

The hyperedges of an $r$-uniform friendship hypergraph are partitioned in $K_{r+1}^r$.

Sós: Does there exist a 3-uniform friendship hypergraph without a universal friend?
Hartke and Vandenbussche, 2008, found 5 friendship hypergraphs without a universal friend:

- $\mathcal{F}^8$: 8 vertices, 8 quads
- $\mathcal{F}^{16}_1$: 16 vertices, 52 quads
- $\mathcal{F}^{16}_2$: 16 vertices, 56 quads
- $\mathcal{F}^{16}_3$: 16 vertices, 68 quads
- $\mathcal{F}^{32}$: 32 vertices, 344 quads
Hartke and Vandenbussche, 2008: The list is complete up to 10 vertices.

Li, van Rees, Seo and Singhi, 2012: The list is complete up to 12 vertices.

**Theorem** Li and van Rees, 2013

Number of hyperedges in friendship hypergraph with a universal friend and with \( n \) vertices = \[
\frac{2}{3}(n - 1)(n - 2)
\]

\( \leq \) number of edges in friendship hypergraph without a universal friend.
The lattice graph $L_2(4)$ is a strongly regular graph.
The 68 quads of $\mathcal{F}_3^{16}$ are:

- a row of $L_2(4)$  
  4 quads
- a column of $L_2(4)$  
  4 quads
- the vertices of a rectangle of $L_2(4)$  
  36 quads
- a transversal of $L_2(4)$  
  24 quads

Two adjacent vertices in $L_2(4)$ are in 4 quads. Two non-adjacent vertices are in 3 quads.
Consider an association scheme \((X, \{R_0, R_1, \ldots, R_s\})\), where \(R_0\) is the identity relation.

A collection of blocks, i.e., \(k\)-subsets of \(X\) is called a Partially Balanced Design if there exist numbers \(\lambda_1, \ldots, \lambda_s\) so that if \((x, y) \in R_i\) then there are exactly \(\lambda_i\) blocks containing \(x\) and \(y\).

All known 3-uniform friendship hypergraphs without a universal friend are partially balanced designs.
(But it is necessary to allow \(\lambda_i = \lambda_j\) when \(i \neq j\).)

And all known 3-uniform friendship hypergraphs without a universal friend are vertex transitive.
**Theorem** J. and S.
Let $X$ be the vertex set of a hypercube of dimension $k$ (Hamming scheme)
(i.e. the set of bitstrings of length $k$).

Then hypergraph with vertex set $X$
where $\{x, y, z\}$ is a hyperedge if

$$\text{dist}(x, y) + \text{dist}(x, z) + \text{dist}(y, z) = 2^k$$

is a 3-uniform friendship hypergraph
with $2^k$ vertices and $2^{k-3}(3^{k-1} - 1)$ quads.

For $k = 3$: $\mathcal{F}^8$
For $k = 4$: $\mathcal{F}_1^{16}$
Proof

Let \( x, y, z \in X \).
We may assume that

\[
x = 0 \ldots 0 0 \ldots 0 0 \ldots 0 0 \ldots 0
\]
\[
y = 1 \ldots 1 1 \ldots 1 0 \ldots 0 0 \ldots 0
\]
\[
z = 1 \ldots 1 0 \ldots 0 1 \ldots 1 0 \ldots 0
\]

Then

\[
w = 0 \ldots 0 1 \ldots 1 1 \ldots 1 1 \ldots 1
\]

is the unique vertex satisfying that

\[
\{x, y, w\}, \{x, z, w\}, \{y, z, w\}
\]

are hyperedges.
Theorem J. and S.

There are three non-isomorphic 3-uniform friendship hypergraphs on 20 vertices with regular group of automorphisms. Each of them is cyclic and has 105 quads.

Theorem J. and S.

There is at least one 3-uniform friendship hypergraph on 28 vertices with a cyclic regular group of automorphisms. It has 259 quads.
Some open problems on 3-uniform friendship hypergraphs with \( n \) vertices and without universal friend:

- Is every vertex in the same number of hyperedges?
- Is every friendship hypergraph a partially balanced design?
- Is \( n \) always a multiple of 4?
- Is every vertex the completion of \( \binom{n}{3}/n = \frac{1}{6}(n - 1)(n - 2) \) 3-sets?
- Does there exist a friendship hypergraph with \( n \equiv 0 \pmod{3} \)?
• Does there exist a friendship hypergraph for every $n \equiv 4$ or $8$ (mod 12)?
**Theorem**  J. and S.

The 5–(12, 6, 1) design has three points a, b, c and the 9 vertices of the hypergraph.

The hyperedges are 4-subsets of $B - a$, for block $B$ containing a but not b or c.

This is a 4-uniform friendship hypergraph.
References


